Multilevel Solvers for Discontinuous Galerkin Methods

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Main Objective:

Design and analyze efficient solution techniques for $hp-$discontinous Galerkin discretizations of

$$-\nabla \cdot (\mathbf{K} \nabla u) = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \partial\Omega.$$

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where the bilinear form a_ϵ is given by

$$a_\epsilon(u_h, v) = \sum_{E \in \mathcal{E}_h} \int_E \mathbf{K} \nabla u_h \cdot \nabla v - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\{\mathbf{K} \nabla u_h \cdot \vec{n}_e\}\} [\![v]\!]$$
$$+ \epsilon \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\{\mathbf{K} \nabla v \cdot \vec{n}_e\}\} [\![u_h]\!] + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{\sigma_e}{|e|^{\beta_0}} \int_e [\![u_h]\!] [\![v]\!].$$

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To solve this system we employ a Schwarz domain decomposition method:

$$B = \mathbf{R}_0 \mathbf{A}_0^{-1} \mathbf{R}_0^T + \sum_{j=1}^{N_S} \mathbf{R}_j \mathbf{A}_j^{-1} \mathbf{R}_j^T$$

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R_j are elliptic projection operators, and $\mathbf{A}_j$ are the restriction of $a_\epsilon(\cdot,\cdot)$ to a subdomain Ω_j.

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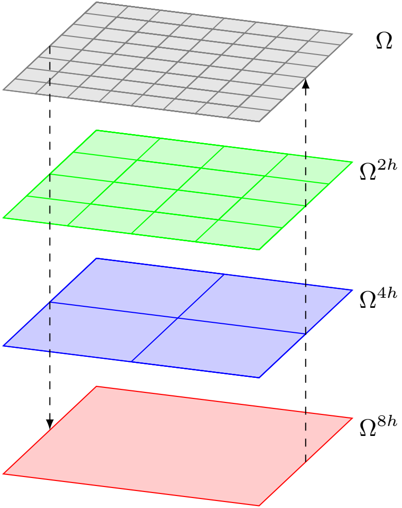

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- The **fine** grid operators $(\mathbf{R}_j \mathbf{A}_j^{-1} \mathbf{R}_j^T)$ are amenable to **fine** grain parallelism

Hybrid computing strategy

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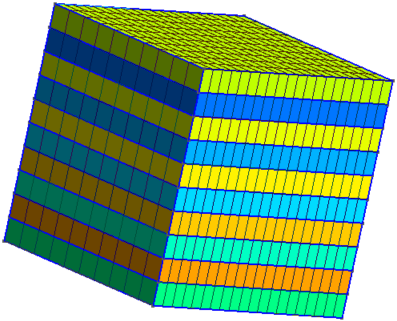

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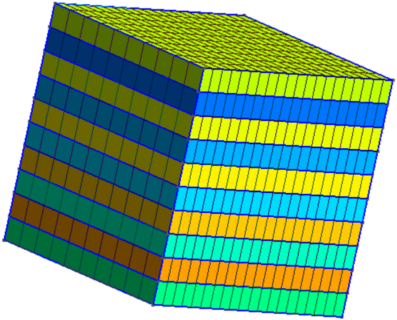

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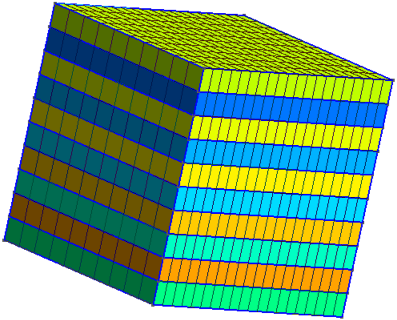

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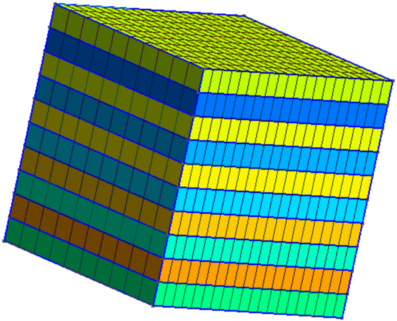

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- Coarse grid is handled by a small number of MPI ranks

Closer look at mapping to coprocessor

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