# Chapter 1: Coloring Mathematically

## Section 1

## Coloring Pictures and Maps

Imagine that you are an illustrator for a children's literature book. The author is very particular about the pictures in her book and would like them to convey a clear, crisp image of her story. She explains, "If two neighboring parts of a picture are colored with the same color, you won't be able to tell where one part ends and the other begins, so I would like any two parts that touch each other to have different colors." Can you fulfill her request?

One of her pictures is the bear in Figure 1. If you count the parts of the picture, you will find that there are eleven regions, counting the two parts of the band and the bow tie as three separate regions. You could, therefore, fulfill her request if you had eleven different colors; you could just use a different color for each region.

However, every new color incurs an additional charge, so if you use fewer colors you can keep production costs to a minimum. As a result, you decide to fulfill her request using as few colors as possible.

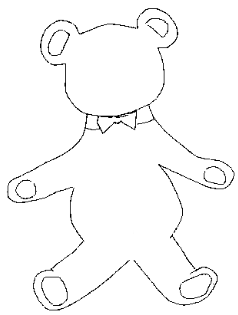

**Figure 1.1**

**Terminology:**
**Coloring ...**

To avoid repeating the phrase "neighboring parts have different colors" over and over, we will adopt the understanding that in a **coloring** of a picture, neighboring parts always have different colors, and when you are asked to **color** a picture, you must always color neighboring parts with different colors. You may object to this requirement and say that there are times when you might actually want to color two regions using the same color. That's okay – feel free to do so. However, in discussing coloring from a mathematical perspective it is useful to insist that neighboring regions always have different colors. As a result, we will use the words "color" and "coloring" differently from how they are used in ordinary English. Mathematicians have their own special language, much like people in other professions and cultures, and use words like "color" and "coloring" in their own special way.

### Activity 1

**An Illustrator's Dilemma**

(**Note:** Each numbered Activity can be found in the *Activity Workbook*.)

Color the eleven regions of the bear using as few colors as possible. Remember that neighboring parts must have different colors (see the terminology note in the side column). How many colors did you use? How do you know that you can't color the bear using fewer colors?

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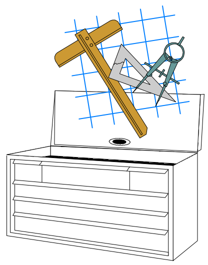
**Go to the Activity Book now, before reading any further, and complete Activity 1.**

One of the key features of this activity, and of this book in general, is problem solving. Initially, when asked to color Figure 1, most people (adults or children) would not naturally take a mathematical approach, but would more likely choose colors on the basis of what seems attractive, or might choose colors more or less randomly. But when asked to



**Set Goals and Subgoals!**

When asked to color a picture using as few colors as possible, your initial response might be "I don't have a clue how many colors are needed". An important strategy in mathematical problem solving is setting a goal. Indeed, setting goals is often the key to solving a problem. Sometimes the goal is clear – coloring a picture using five colors, for example. Sometimes the goal is less explicit – color a picture using a minimum number of colors, for example, as in the current problem. In cases like this where the goal is broad, you may need to break it down into smaller, more manageable pieces, called subgoals. For example, if you start off with a coloring that uses six colors, you might first try to reduce the number of colors to five, then to four, etc. Stating goals and, if necessary, subgoals can give focus to your efforts – they enable you to know what it is that you are trying to accomplish.

**Figure 1.2**

color Figure 1 using as few colors as possible, they might shift into a problem-solving mode, since they were now presented with a "problem", that is, a situation where a path to a solution is not evident.

Before you try to solve a problem, you need to clarify what it is that you are trying to do. That is, an initial step in solving the problem is stating explicitly your goal. In this problem, the goal is to use as few colors as possible to color this picture so that neighboring parts have different colors. How might you achieve this goal if you started out with a coloring that used many colors (perhaps even eleven colors, one for each region)? You might try to color the picture using one fewer color, then one fewer than that, etc., until you could no longer reduce the number. These steps in solving the problem could be considered subgoals – more modest goals that together help you achieve your overall goal. (See note in the side column.) To achieve the subgoal of using one fewer color, you might look at a region and simply change its color to one that was already used for another region.

This strategy involves starting with a fixed number of colors, and reducing the number one at a time. Another strategy is to go in the opposite direction, starting with one color and adding in colors, one at a time. Here the subgoals are to hold the number of colors used to one, then to two, then to three, etc. That is, you try to achieve the "minimum color" goal by first trying to color the picture using only two colors. If that works, great! If not, you'll try to color the picture using the next fewest colors, that is, three colors. If that works, great! If not, you'll try to color the picture using four colors. If that works, great! If not, you'll try to do it using five colors. Eventually you will determine how many colors are needed, even though at no point did you know what the answer would be. We will refer to this strategy as the "stingy strategy", since you only use additional colors when you really have to.

If you followed either of these strategies, you probably came to the conclusion that it was possible to color the picture using three colors (as in Figure 1.2), and that it didn't seem possible to color the picture using two colors. However, to convert the intuitive "I *think* that it can't be done using two colors" to a definite "I *know* that it can't be done using two colors" requires *reasoning*, where you explain why you arrived at a certain conclusion. A frequent response to "Why?" is "just because." Problem solvers need to be able to put into words their "just because" response, so that they can convince others that their understanding of the situation is correct.

Why does this picture require three colors? Since the bow of the bow tie and the band of the bow tie are touching, they must be different colors – say green and pink. But the bear's head (or torso) touches both of them, so it can't be

**Figure 1.3**

green or pink, and must be a third color, for example, brown. That's why three colors are needed. This simple reasoning provides a concrete explanation that is readily understood and that justifies why three colors are necessary.

We can color maps as well as pictures. For example, what is the smallest number of colors needed for the forty-eight states in the continental United States? Of course, as in the previous activity, any two states that share a common border must have different colors.

## Activity 2

### Coloring the Map of the Western States

**(Note:** Each numbered Activity can be found in the *Activity Workbook*.)

Can you color the map of the twenty-two states west of the Mississipi River (See Figure 3) using a small number of colors? Remember, if any two states share a border, they must have different colors.

STOP

**Go to the Activity Book now, before reading any further, and complete Activity 2.**

Coloring the United States map is a challenging problem. You may have found that it was possible to color the twenty-two western states using four colors, but try as you might you couldn't do it using three colors. Why can't you color this map using three colors? At the moment, that question appears difficult to answer.

One strategy that you may have learned for solving problems that appear difficult is 'Try a simpler problem!" (See note in side column.) In coloring the United States map, we would have to deal with 48 states. A simpler problem involving 22 states appears in Activity 2. But that problem also appears difficult, so we look at a problem involving fewer states. Activity 3 involves four regions of the United States, each with four to seven states. Once we discuss this activity, we will see why we found it difficult to color the states west of the Mississippi River using three colors.

What good problem solvers learn to do is ask themselves automatically: 'Can I first solve a simpler related problem?" Can you color the United States map using three colors? Let us try a similar task on some smaller regions of the map. 'Try a simpler problem!" doesn't always work, but

**Try a Simpler Problem!**

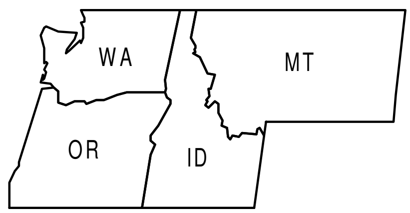
"Try a Simpler Problem!" is one of the dozen-or-so important problem-solving strategies. " Try a Simpler Problem!" can be used whenever you are stuck solving a complex problem but can formulate simpler versions of the problem; solving the simpler problems will often give you insight into solving the original problem. For example, you want to know how many thingamajigs there are in fifteen widgets – so first determine how many there are in one, two, and three widgets. Maybe the solutions to these simpler problems will give you a clue or provide insight to a pattern that will help you solve the original problem of how many thingamajigs there are in fifteen widgets. Here's a complex problem: "Can you color the states west of the Mississippi River using only three colors?" We will try coloring some smaller portions of the United States map to see if we can gain insight into the solution!

it is one of the first problem-solving strategies you should think of when you don't know how to proceed.

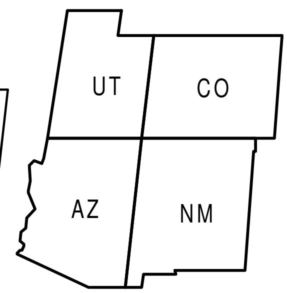
## Activity 3

### Coloring Four Regions of the United States

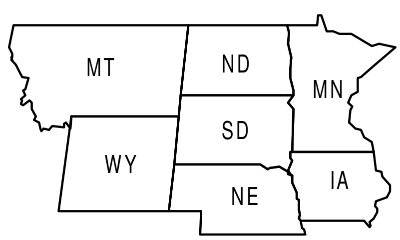
Can you color the states in each of Figures 4, 5, 6, and 7 using three colors – say red, white, and blue? Remember that any two states that share a common border must be colored using different colors so that you can tell where one state ends and the other begins. If you can't color a map using three colors, what is the fewest number of colors needed?



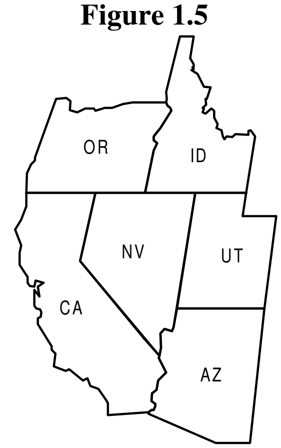

**Map A**
**Figure 1.4**




**Map B**
**Figure 1.5**




**Map C**
**Figure 1.6**

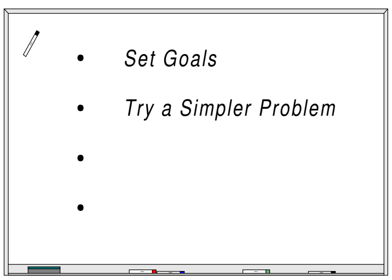



**Map D**
**Figure 1.7**


**Go to the Activity Book now, before reading any further, and complete Activity 3.**

**The Mathematician's Toolbox**

Like a carpenter with a set of tools, the mathematician, when faced with a new situation, looks into the toolbox and decides which tool seems most appropriate. The answer to the question "What can I do?" is "Look into your toolbox, and try out the tool that seems most appropriate."




In the toolbox above, pictured as a blackboard, we have put the two tools we have already discussed. As we proceed, more tools will be added to this toolbox.